Real Analysis - I

theorem:—

Every convergent sequence is bounded.

proof:—

Let $\{S_n\}$ be a convergent sequence and its limit is l

Given $\varepsilon = 1 > 0$ $\exists\, m \in N$ such that

$$|s_n - l| < \varepsilon = 1 \ \forall\, n \geq m$$

$$\Rightarrow |s_n - l| < 1, \forall\, n \geq m$$

$$\Rightarrow -1 < s_n - l < 1, \forall\, n \geq m$$

$$\Rightarrow l - 1 < s_n < l + 1, \forall\, n \geq m$$

$k_1 = \min \{l-1, s_1, s_2 \ldots\ldots s_m\}$

$k_2 = \max \{l+1, s_1, s_2 \ldots\ldots s_m\}$

then $k_1 \leq s_n \leq k_2 \ \forall\, n \in N$

$\{s_n\}$ is bounded

Q. show that $\lim\limits_{n\to\infty} n^{1/n} = 1$

$s_n = n^{1/n} = 1$ for $n > 1$

$(1 + s_n) = n^{1/n}$

$(1 + s_n)^n = n$

$$n = (1+s_n)^n$$

$$= nC_0\, 1^n + nC_1\, 1^{n-1} s_n + nC_2\, 1^{n-2} s_n^2 + \cdots + nC_n s_n^n$$

$$= 1 + n 1 s_n + \frac{n(n-1)}{2!} 1 \cdot s_n^2 + \cdots\cdots + s_n^n$$

$$= 1 + n s_n + \frac{n(n-1)}{2} s_n^2 + \cdots + s_n^n > \frac{n(n-1)}{2} s_n^2$$

$$= n > \frac{n(n-1)}{2} s_n^2$$

$\Rightarrow \frac{2n}{n(n-1)} > \delta_n^2$

$\Rightarrow \delta^2_n < \frac{2}{n-1}$

$\Rightarrow \delta_n < \sqrt{\frac{2}{n-1}}$

Given $\varepsilon > 0$ choose n such that

$\sqrt{\frac{2}{n-1}} < \varepsilon \Longleftrightarrow \frac{2}{n-1} < \varepsilon^2 \Longleftrightarrow \frac{2}{\varepsilon^2} < n-1$

$\Longleftrightarrow 1 + \frac{2}{\varepsilon^2} < n$

choose $m \in N \ni 1 + \frac{2}{\varepsilon^2} < m$

then $\delta_n < \sqrt{\frac{2}{n-1}} < \varepsilon \ \forall n \geq m$

$\Rightarrow \lim_{n \to \infty} \delta_n = 0$

$\lim_{n \to \infty} (n^{1/n} - 1) = 0$

$\Rightarrow \lim_{n \to \infty} n^{1/n} = 1$

√Imp theorem: —

Cauchy's first theorem on limits.

pre 10M. statement: —

$\lim_{n \to \infty} \delta_n = l$ then $\lim_{n \to \infty} \frac{\delta_1 + \delta_2 + \cdots + \delta_n}{n}$

proof: —

Define $t_n = \delta_n - l$

$\lim_{n \to \infty} t_n = \lim_{n \to \infty} (\delta_n - l)$

$= l - l$

Given $\varepsilon > 0 \ \exists \ m \in N \ni |t_n| < \frac{\varepsilon}{2} \ \forall n \geq m$

since $\{t_n\}$ is convergent it is bounded

$\Rightarrow |t_n| < k \ \forall n$

$\left| \frac{t_1 + t_2 + \cdots + t_n + t_{n+1} + \cdots t_n}{n} \right| \leq \frac{|t_1| + |t_2| + \cdots + |t_n| + |t_{n+1}| + \cdots + |t_n|}{n}$

$\leq \frac{k + k + \cdots + k}{n} + \frac{\frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2}}{n}$

$$\le \frac{mk}{n} + \frac{(n-m)}{n}\frac{\varepsilon}{2}$$

$$\le \frac{mk}{n} + \left(1-\frac{m}{n}\right)\frac{\varepsilon}{2}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}$$

$\exists\, p \in \mathbb{N} \ni \frac{mk}{n} < \frac{\varepsilon}{2}\ \forall\, n \ge p$

$m_1 = \max\{m, p\}$

then

$$\left|\frac{t_1+t_2+\cdots+t_n}{n}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,\ \forall\, n \ge m_1$$

$$\Rightarrow \lim_{n\to\infty} \frac{t_1+t_2+\cdots+t_n}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{s_1-l+s_2-l+\cdots+s_n-l}{n}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{s_1+s_2+\cdots+s_n}{n} - \frac{nl}{n}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{s_1+s_2+\cdots+s_n}{n} - l\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{s_1+s_2+\cdots+s_n}{n} = l$$

P Discuss the nature of the sequence $\{r^n\}$ for all $r \in \mathbb{R}$.

pyq
10M

Case-i :—

let $r > 1$

$h = r-1 > 0$

$r = 1+h$

$r^n = (1+h)^n$

$= 1 + nh + \frac{n(n-1)}{2} n^2 + \cdots + h^n > nh.$

$\Rightarrow r^n > nh$

for $k > 0$ choose m such that $nh > k\ \forall\, n \ge m$

$\Rightarrow nh \to \infty$ as $n \to \infty$

Case-ii :— $r = 1$

$r^n = 1^n$

$= 1$

$\therefore r^n = 1$ as $n \to \infty$

case - iii : —

$$0<r<1$$

$$\frac{1}{r}>1$$

let $h=\frac{1}{r}-1>0$

$$1+h=\frac{1}{r}$$

$$\frac{1}{1+h}=r$$

$$r^n=\frac{1}{(1+h)^n}$$

$$r^n=\frac{1}{1+nh+\frac{n(n-1)}{2}h^2+\cdots+h^n}<\frac{1}{nh}$$

$$0<r^n<\frac{1}{nh}$$

Given $\varepsilon>0$ $\exists m\in\mathbb{N}$ such that $\frac{1}{nh}<\varepsilon$ $\forall n\geq m$

$\Rightarrow \frac{1}{nh}\to 0$ as $n\to\infty$

$\Rightarrow r^n\to 0$ as $n\to\infty$

case - iv : —

$$r=0$$

$$r^n=0^n$$

$$=0$$

$\Rightarrow r^n\to 0$ as $n\to\infty$

case - v : —

$$-1<r<0$$

$$s=-r\to -s=r$$

$$-1<-s<0$$

$$1>s>0$$

From case - iii : —

$$s^n\to 0 \text{ as } n\to\infty$$

$$r^n=(-s)^n$$

$$=(-1)^n s^n$$

$(-1)^n$ is bounded sequence

$$s^n\to 0 \text{ as } n\to\infty$$

$$r^n\to 0 \text{ as } n\to\infty$$

case-vi :—

$$r = -1$$

$$r^n = (-1)^n$$

$$= \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

case-vii :—

$$r < -1$$

$$s = -r \Rightarrow r = -s$$

$$-s < -1 \Rightarrow s > 1$$

From case-i :—

$$s^n \to \infty \text{ as } n \to \infty$$

$$r^n = (-s)^n$$

$$r^n = \begin{cases} s^n & \text{if } n \text{ is even} \\ -s^n & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow r^n$ is oscillator

In summary :—

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } -1 < r < 1$$

$$\lim_{n\to\infty} r^n = 1 \quad \text{if } r = 1$$

In all other cases $\{r^n\}$ diverges.

Monotonic sequence :—

Monotonically increasing :—

A sequence $\{s_n\}$ is said to be monotonically increasing if

$$s_m \le s_n \quad \text{if } m \le n$$

Eg :— $\{s_n\} = \{n^2\}$

$\{s_n\} = \{2n^2 - 1\}$

Monotonically decreasing :—

A sequence $\{s_n\}$ is said to be monotonically decreasing if

$$s_m \ge s_n \quad \text{if } m \le n$$

Eg :— $\{S_n\} = \{\frac{1}{n}\}$

$\{S_n\} = \{\frac{1}{2n+1}\}$

Monotone convergence theorem :—

Monotonic sequence is convergent iff it is bou[nded]

Proof :—

let $\{S_n\}$ be a monotonic sequence.

case-i :—

Assume $\{S_n\}$ is convergent to limit l.

Given $\varepsilon = 1 > 0$ there exists $n \in N \ni$

$|S_n - l| < \varepsilon = 1, \forall n \geq m$

$\Rightarrow |S_n - l| < 1, \forall n \geq m$

$\Rightarrow -1 < S_n - l < 1, \forall n \geq m$

$\Rightarrow l - 1 < S_n < l + 1, \forall n \geq m$

take $k_1 = \min\{l-1, S_1, S_2 --- S_m\}$

$k_2 = \max\{l+1, S_1, S_2 ---- S_m\}$

$\Rightarrow k_1 \leq S_n \leq k_2, \forall n \in N.$

$\{S_n\}$ is bounded.

case-ii :—

Assume $\{S_n\}$ is bounded monotonic sequence

a) If $\{S_n\}$ is monotonically increasing.

Assume l is lub to $\{S_n\}$

$S_n \leq l \;\forall n \in N.$

Given $\varepsilon > 0$, $l - \varepsilon < l$ and

$l - \varepsilon$ is not an upper bound to $\{S_n\}$

$\exists m \in N \ni l - \varepsilon < S_m \leq S_{m+1} .... < l$

or

$\exists m \in N \ni l - \varepsilon < S_m < l \;\forall n \geq m$

$\Rightarrow l - \varepsilon < S_n < l < l + \varepsilon, \forall n \geq m$

$\Rightarrow -\varepsilon < S_n - l < \varepsilon, \forall n \geq m$

$\Rightarrow |S_n - l| < \varepsilon, \forall n \geq m$

that is $\lim_{n\to\infty} s_n = l$

$\therefore \{s_n\}$ is convergent

(b) If $\{s_n\}$ is monotonically decreasing and bounded then $\{t_n\} = \{-s_n\}$ is monotonically increasing and bounded from (a) $\{t_n\}$ is convergent say its limit is l.

$\Rightarrow \lim_{n\to\infty} t_n = l$

$\Rightarrow \lim_{n\to\infty} (-s_n) = l$

$\Rightarrow \lim_{n\to\infty} s_n = l$

$\Rightarrow \lim_{n\to\infty} s_n = -l$

$\therefore \{s_n\}$ is convergent.

P If $s_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$ then $\{s_n\}$ is convergent.

$$s_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} + \frac{1}{n+n+1} + \frac{1}{n+n+2}$$

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

$$s_{n+1} - s_n = \frac{1}{n+n+1} + \frac{1}{n+n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{n+1-2n-2}{(2n+2)(n+1)}$$

$$= \frac{1}{2n+1} + \frac{-n-1}{(2n+2)(n+1)}$$

$$= \frac{1}{2n+1} + \frac{-(\cancel{n+1})}{(2n+2)(\cancel{n+1})}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2}$$

$= \dfrac{2n+2-2n-1}{(2n+1)(2n+2)}$

$= \dfrac{1}{(2n+1)(2n+2)} > 0$

$s_{n+1} - s_n > 0$

$s_{n+1} > s_n$

$\therefore \{s_n\}$ is monotonically increasing.

$s_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{n+n} < \dfrac{1}{n} + \dfrac{1}{n} + \cdots + \dfrac{1}{n} = n \cdot \dfrac{1}{n}$

$s_n < 1 \ \forall n \in N$

$\Rightarrow \{s_n\}$ is increasing and bounded.

$\Rightarrow \{s_n\}$ is convergent.

I If $s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$ then $\{s_n\}$ is convergent.

$s_{n+1} = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

$s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}$

$s_{n+1} - s_n = \dfrac{1}{(n+1)!} > 0$

$\Rightarrow s_{n+1} > s_n$

$\Rightarrow \{s_n\}$ is monotonically increasing.

$s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

$< 1 + 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^{n-1}}$

$s_n < 1 + \dfrac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 1 + \dfrac{1 - \frac{1}{2^n}}{\frac{1}{2}}$

$s_n < 1 + 2\left(1 - \dfrac{1}{2^n}\right) = 1 + 2 - \dfrac{2}{2^n} < 3$

$\Rightarrow s_n < 3$

$\therefore \{s_n\}$ is monotonically increasing and bounded above.

$\Rightarrow \{s_n\}$ is convergent.

Cauchy's sequence :—

Let $\{s_n\}$ be a sequence it is said to be cauchy if given $\varepsilon>0 \; \exists \; m\in N$. such that $|s_p - s_q| < \varepsilon \; \forall$
$p>q\geq m$.

theorem :—

A sequence is convergent iff it is cauchy.

proof :—

Let $\{s_n\}$ be a sequence ~~is convergent to limit l.~~

case-i :—

If $\{s_n\}$ is convergent to limit l.

Given $\varepsilon>0 \; \exists \; m\in N \; \exists \; |s_n - l| < \frac{\varepsilon}{2}, \forall \; n\geq m$.

$p>q\geq m \Rightarrow |s_p - l| < \frac{\varepsilon}{2}$ and $|s_q - l| < \frac{\varepsilon}{2}$

$|s_p - s_q| = |s_p - l + l - s_q| \leq |s_p - l| + |l - s_q|$

$|s_p - s_q| \leq |s_p - l| + |s_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall \; p>q\geq m$

$\{s_n\}$ is convergent.

$\therefore \{s_n\}$ is cauchy

case-ii :—

Assume $\{s_n\}$ is cauchy sequence

Given $\varepsilon = 1>0 \; \exists \; m\in N \; \exists \; |s_p - s_q| < \varepsilon = 1 \; \forall \; p>q\geq m$

$|s_p - s_q| < 1, \forall \; p>q\geq m$

$|s_p - s_m| < 1, \forall \; p>m$

$-1 < s_p - s_m < 1 \; \forall \; p>m$

$s_m - 1 < s_p < s_m + 1$

$k_1 = \min\{s_m - 1, s_1, s_2 - \cdots s_m\}$

$k_2 = \max\{s_m + 1, s_1, s_2 - \cdots s_m\}$

$k_1 \leq s_n \leq k_2$.

$\therefore \{s_n\}$ is bounded.

$\Rightarrow$ we have a bounded sequence $\{s_n\}$

By Bolzano-weistrass theorem $\exists$ a limit

point say l to $\{s_n\}$.
If there is another limit point say l' to $\{s_n\}$

$$\varepsilon = |l - l'| > 0$$

since $\{s_n\}$ is cauchy and l, l' are limit points

$\exists\, m \in N \ni$

$$|s_p - s_q| < \frac{\varepsilon}{3}, \ \forall\, p > q \geq m$$

$$|s_p - l| < \frac{\varepsilon}{3}, \ \forall\, p \geq m$$

$$|s_q - l'| < \frac{\varepsilon}{3}, \ \forall\, q \geq m$$

$$|l - l'| = |l - s_p + s_p - s_q + s_q - l'|$$

$$\leq |l - s_p| + |s_p - s_q| + |s_q + l'|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$$|l - l'| < \varepsilon = |l - l'|$$

this is contradiction

$\therefore \{s_n\}$ has only one limit point.

Hence $\{s_n\}$ is convergent.

Pre SM Show that

$$\lim_{n \to \infty} \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$$

$$0 \leq s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$$

$$\leq \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} = \frac{n}{n^2}$$

$$0 \leq s_n \leq \frac{1}{n}$$

$$\lim 0 \leq \lim s_n \leq \lim \frac{1}{n}$$

$$0 \leq \lim s_n \leq 0$$

$$\lim_{n \to \infty} s_n = 0$$

Sandwich theorem:-

If $\{s_n\}, \{t_n\}, \{u_n\}$ are three sequences such that
(i) For some positive integer k, $s_n \le u_n \le t_n, \forall n \in N, n \ge k$
(ii) $\lim s_n = \lim t_n = l$. then $\lim u_n = l$.

proof:-

Let $\varepsilon > 0$

since $\lim s_n = l, \exists m_1 \in N \ni n \in N, n \ge m_1 \Rightarrow$
$|s_n - l| < \varepsilon \Rightarrow l - \varepsilon < s_n < l + \varepsilon$

since $\lim t_n = l, \exists m_2 \in N \ni n \in N, n \ge m_2 \Rightarrow$
$|t_n - l| < \varepsilon \Rightarrow l - \varepsilon < t_n < l + \varepsilon$

Let $m = \max\{m_1, m_2, k\}$. then $n \in N, n \ge m \Rightarrow$
$n \ge m_1, n \ge m_2, n \ge k$.
$\Rightarrow l - \varepsilon < s_n < l + \varepsilon, l - \varepsilon < t_n < l + \varepsilon, s_n \le u_n < t_n \Rightarrow$
$l - \varepsilon < s_n \le u_n \le t_n < l + \varepsilon$
$\Rightarrow l - \varepsilon < u_n < l + \varepsilon \Rightarrow |u_n - l| < \varepsilon$ $\quad \therefore u_n = l$.

$$\sum_1^\infty \frac{1}{2^n + 3^n}$$

let $u_n = \frac{1}{2^n + 3^n}, \forall n$

$= \frac{1}{3^n}\left(\frac{1}{1 + \left(\frac{2}{3}\right)^n}\right)$ $\quad$ Take $v_n = \frac{1}{3^n}$.

It is an I.G.P

By comparision test $\quad$ since $r = 3 > 1$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1$$

Hence $\sum u_n$ also converges.

If $S_n = \left[1+\frac{1}{n}\right]^n$ then show that $\{S_n\}$ is convergent.

If $S_n = \left[1+\frac{1}{n}\right]^n$.

By the binomial theorem.

$$S_n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{n-1}{n}\right)$$

Similarly,

$$S_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) \cdots \left(1-\frac{n}{n+1}\right)$$

$$> 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right) + \cdots + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{n-1}{n}\right)$$

$\therefore S_{n+1} > S_n$ for all $n \in N$.

Hence $\{S_n\}$ is monotonically increasing.

Also, from ①;

$$2 < S_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$$

Real Analysis - I

theorem:—

Every convergent sequence is bounded.

proof:—

Let $\{s_n\}$ be a convergent sequence and its limit is l

Given $\varepsilon = 1 > 0$ $\exists\, m \in N$ such that

$$|s_n - l| < \varepsilon = 1 \ \forall\, n \geq m$$

$$\Rightarrow |s_n - l| < 1, \forall\, n \geq m$$

$$\Rightarrow -1 < s_n - l < 1, \forall\, n \geq m$$

$$\Rightarrow l - 1 < s_n < l + 1, \forall\, n \geq m$$

$k_1 = \min \{l-1, s_1, s_2 \ldots\ldots s_m\}$

$k_2 = \max \{l+1, s_1, s_2 \ldots\ldots s_m\}$

then $k_1 \leq s_n \leq k_2 \ \forall\, n \in N$

$\{s_n\}$ is bounded

$\frac{S_n - l(0)}{n-0}$

Af

Q. Show that $\lim\limits_{n \to \infty} n^{1/n} = 1$

$s_n = n^{1/n} = 1$ for $n > 1$

$(1 + s_n) = n^{1/n}$

$(1 + s_n)^n = n$

$$n = (1 + s_n)^n$$

$$= nC_0\, 1^n + nC_1\, 1^{n-1} s_n + nC_2\, 1^{n-2} s_n^2 + \ldots + nC_n s_n^n$$

$$= 1 + n1 s_n + \frac{n(n-1)}{2!} 1 \cdot s_n^2 + \ldots\ldots + s_n^n$$

$$= 1 + n s_n + \frac{n(n-1)}{2} s^2 n + \ldots + s^n n > \frac{n(n-1)}{2} s_n^2$$

$$= n > \frac{n(n-1)}{2} s_n^2$$

$\Rightarrow \frac{2n}{n(n-1)} > s_n^2$

$\Rightarrow s_n^2 < \frac{2}{n-1}$

$\Rightarrow s_n < \sqrt{\frac{2}{n-1}}$

Given $\varepsilon > 0$ choose n such that

$\sqrt{\frac{2}{n-1}} < \varepsilon \iff \frac{2}{n-1} < \varepsilon^2 \iff \frac{2}{\varepsilon^2} < n-1$

$\iff 1 + \frac{2}{\varepsilon^2} < n$

choose $m \in \mathbb{N} \ni 1 + \frac{2}{\varepsilon^2} < m$

then $s_n < \sqrt{\frac{2}{n-1}} < \varepsilon \ \forall n \geq m$

$\Rightarrow \lim_{n\to\infty} s_n = 0$

~~$n \to \infty$~~

$\lim_{n\to\infty} (n^{1/n} - 1) = 0$

$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1$

√Imp **theorem**: —

Cauchy's first theorem on limits.

pre 10M **statement**: —

$\lim_{n\to\infty} s_n = l$ then $\lim_{n\to\infty} \frac{s_1 + s_2 + \cdots + s_n}{n}$

Proof: —

Define $t_n = s_n - l$

$\lim_{n\to\infty} t_n = \lim_{n\to\infty} (s_n - l)$

$= l - l$

Given $\varepsilon > 0 \ \exists m \in \mathbb{N} \ni |t_n| < \frac{\varepsilon}{2} \ \forall n \geq m$

since $\{t_n\}$ is convergent it is bounded

$\Rightarrow |t_n| < k \ \forall n$

$\left|\frac{t_1 + t_2 + \cdots + t_m + t_{m+1} + \cdots t_n}{n}\right| \leq \frac{|t_1| + |t_2| + \cdots + |t_m| + |t_{m+1}| + \cdots + |t_n|}{n}$

$\leq \frac{k + k + \cdots + k}{n} + \frac{\frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2}}{n}$

$$\leq \frac{mk}{n} + \frac{(n-m)}{n}\frac{\varepsilon}{2}$$

$$\leq \frac{mk}{n} + \left(1-\frac{m}{n}\right)\frac{\varepsilon}{2}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}$$

$\exists\, p \in \mathbb{N} \ni \frac{mk}{n} < \frac{\varepsilon}{2}\ \forall n \geq p$

$m_1 = \max\{m, p\}$

then

$$\frac{|t_1+t_2+\cdots+t_n|}{n} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,\ \forall n \geq m_1$$

$$\Rightarrow \lim_{n\to\infty} \frac{t_1+t_2+\cdots+t_n}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{s_1-l+s_2-l+\cdots+s_n-l}{n}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{s_1+s_2+\cdots+s_n}{n} - \frac{nl}{n}\right] = 0$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{s_1+s_2+\cdots+s_n}{n} - l\right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{s_1+s_2+\cdots+s_n}{n} = l$$

P Discuss the nature of the sequence $\{r^n\}$ for all $r \in \mathbb{R}$.

Case-i :—

let $r>1$

$h = r-1 > 0$

$r = 1+h$

$r^n = (1+h)^n$

$= 1+nh+\frac{n(n-1)}{2}h^2+\cdots+h^n > nh.$

$\Rightarrow r^n > nh$

for $k>0$ choose m such that $nh > k\ \forall n \geq m$

$\Rightarrow nh \to \infty$ as $n\to\infty$

Case-ii :—

$r=1$

$r^n = 1^n$

$= 1$

$\therefore r^n = 1$ as $n\to\infty$

Case - iii : —

$$0 < r < 1$$

$$\frac{1}{r} > 1$$

let $h = \frac{1}{r} - 1 > 0$

$$1 + h = \frac{1}{r}$$

$$\frac{1}{1+h} = r$$

$$r^n = \frac{1}{(1+h)^n}$$

$$r^n = \frac{1}{1 + nh + \frac{n(n-1)}{2}h^2 + \cdots + h^n} < \frac{1}{nh}$$

$$0 < r^n < \frac{1}{nh}$$

Given $\varepsilon > 0$ $\exists\, m \in \mathbb{N}$ such that $\frac{1}{nh} < \varepsilon$ $\forall n \geq m$

$\Rightarrow \frac{1}{nh} \to 0$ as $n \to \infty$

$\Rightarrow r^n \to 0$ as $n \to \infty$

Case - iv : —

$$r = 0$$

$$r^n = 0^n$$

$$= 0$$

$\Rightarrow r^n \to 0$ as $n \to \infty$

Case - v : —

$$-1 < r < 0$$

$$s = -r \Rightarrow -s = r$$

$$-1 < -s < 0$$

$$1 > s > 0$$

From case - iii : —

$s^n \to 0$ as $n \to \infty$

$$r^n = (-s)^n$$

$$= (-1)^n s^n$$

$(-1)^n$ is bounded sequence

$s^n \to 0$ as $n \to \infty$

$r^n \to 0$ as $n \to \infty$

case-vi : —

$$r = -1$$
$$r^n = (-1)^n$$
$$= \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

case-vii : —

$$r < -1$$
$$s = -r \Rightarrow r = -s$$
$$-s < -1 \Rightarrow s > 1$$

From case-i : —

$$s^n \to \infty \text{ as } n \to \infty$$
$$r^n = (-s)^n$$
$$r^n = \begin{cases} s^n & \text{if } n \text{ is even} \\ -s^n & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow r^n$ is oscillator

In summary : —

$$\lim_{n\to\infty} r^n = 0 \quad \text{if } -1 < r < 1$$
$$\lim_{n\to\infty} r^n = 1 \quad \text{if } r = 1$$

In all other cases $\{r^n\}$ diverges.

Monotonic sequence : —

Monotonically increasing : —

A sequence $\{s_n\}$ is said to be monotonically increasing if

$$s_m \le s_n \quad \text{if } m \le n$$

Eg: — $\{s_n\} = \{n^2\}$

$\{s_n\} = \{2n^2 - 1\}$

Monotonically decreasing : —

A sequence $\{s_n\}$ is said to be monotonically decreasing if

$$s_m \ge s_n \quad \text{if } m \le n$$

Eg :— $\{s_n\} = \{\frac{1}{n}\}$

$\{s_n\} = \{\frac{1}{2n+1}\}$

Monotone convergence theorem :—

Monotonic sequence is convergent iff it is boun[ded]

Proof :—

Let $\{s_n\}$ be a monotonic sequence.

Case-i :—

Assume $\{s_n\}$ is convergent to limit l.

Given $\varepsilon = 1 > 0$ there exists $n \in N \ni$

$|s_n - l| < \varepsilon = 1, \forall n \geq m$

$\Rightarrow |s_n - l| < 1, \forall n \geq m$

$\Rightarrow -1 < s_n - l < 1, \forall n \geq m$

$\Rightarrow l-1 < s_n < l+1, \forall n \geq m$

take $k_1 = \min\{l-1, s_1, s_2 --- s_m\}$

$k_2 = \max\{l+1, s_1, s_2 ---- s_m\}$

$\Rightarrow k_1 \leq s_n \leq k_2, \forall n \in N.$

$\{s_n\}$ is bounded.

Case-ii :—

Assume $\{s_n\}$ is bounded monotonic sequence

a) If $\{s_n\}$ is monotonically increasing.

Assume l is lub to $\{s_n\}$

$s_n \leq l \ \forall n \in N.$

Given $\varepsilon > 0$, $l - \varepsilon < l$ and

$l - \varepsilon$ is not an upper bound to $\{s_n\}$

$\exists m \in N \ni l - \varepsilon < s_m \leq s_{m+1} .... < l$

or

$\exists m \in N \ni l - \varepsilon < s_m < l \ \forall n \geq m$

$\Rightarrow l - \varepsilon < s_n < l < l + \varepsilon, \forall n \geq m$

$\Rightarrow -\varepsilon < s_n - l < \varepsilon, \forall n \geq m$

$\Rightarrow |s_n - l| < \varepsilon, \forall n \geq m$

that is $\lim_{n\to\infty} S_n = l$

$\therefore \{S_n\}$ is convergent

(b) If $\{S_n\}$ is monotonically decreasing and bounded then $\{t_n\} = \{-S_n\}$ is monotonically increasing and bounded from (a) $\{t_n\}$ is convergent say its limit is l.

$\Rightarrow \lim_{n\to\infty} t_n = l$

$\Rightarrow \lim_{n\to\infty} (-S_n) = l$

$\Rightarrow -\lim_{n\to\infty} S_n = l$

$\Rightarrow \lim_{n\to\infty} S_n = -l$

$\therefore \{S_n\}$ is convergent.

P If $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ then $\{S_n\}$ is convergent.

Sol $S_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{n+n} + \frac{1}{n+n+1} + \frac{1}{n+n+2}$

$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$$S_{n+1} - S_n = \frac{1}{n+n+1} + \frac{1}{n+n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} + \frac{n+1-2n-2}{(2n+2)(n+1)}$$

$$= \frac{1}{2n+1} + \frac{-n-1}{(2n+2)(n+1)}$$

$$= \frac{1}{2n+1} + \frac{-(\cancel{n+1})}{(2n+2)(\cancel{n+1})}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= \frac{2n+2-2n-1}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0$$

$s_{n+1} - s_n > 0$

$s_{n+1} > s_n$

$\therefore \{s_n\}$ is monotonically increasing.

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} = \frac{n \cdot 1}{n}$$

$s_n < 1 \; \forall n \in N$

$\Rightarrow \{s_n\}$ is increasing and bounded.

$\Rightarrow \{s_n\}$ is convergent.

I If $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ then $\{s_n\}$ is convergent.

$$s_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$s_{n+1} - s_n = \frac{1}{(n+1)!} > 0$$

$\Rightarrow s_{n+1} > s_n$

$\Rightarrow \{s_n\}$ is monotonically increasing.

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$s_n < 1 + \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 1 + \frac{1 - \frac{1}{2^n}}{\frac{1}{2}}$$

$$s_n < 1 + 2\left(1 - \frac{1}{2^n}\right) = 1 + 2 - \frac{2}{2^n} < 3$$

$\Rightarrow s_n < 3$

$\therefore \{s_n\}$ is monotonically increasing and bounded above.

$\Rightarrow \{S_n\}$ is convergent.

Cauchy's sequence :—

Let $\{S_n\}$ be a sequence it is said to be cauchy if given $\varepsilon > 0$ $\exists$ $m \in \mathbb{N}$. such that $|S_p - S_q| < \varepsilon$ $\forall$ $p > q \geq m$.

theorem :—

A sequence is convergent iff it is cauchy.

proof :—

Let $\{S_n\}$ be a sequence ~~is convergent~~ to limit l.

case-i :—

If $\{S_n\}$ is convergent to limit l.

Given $\varepsilon > 0$ $\exists$ $m \in \mathbb{N}$ $\exists$ $|S_n - l| < \frac{\varepsilon}{2}$, $\forall$ $n \geq m$.

$p > q \geq m \Rightarrow |S_p - l| < \frac{\varepsilon}{2}$ and $|S_q - l| < \frac{\varepsilon}{2}$

$|S_p - S_q| = |S_p - l + l - S_q| \leq |S_p - l| + |l - S_q|$

$|S_p - S_q| \leq |S_p - l| + |S_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\forall$ $p > q \geq m$

$\{$~~S_n~~ is ~~convergent~~.

$\therefore \{S_n\}$ is cauchy

case-ii :—

Assume $\{S_n\}$ is cauchy sequence

Given $\varepsilon = 1 > 0$ $\exists$ $m \in \mathbb{N}$ $\exists$ $|S_p - S_q| < \varepsilon = 1$ $\forall$ $p > q \geq m$

$|S_p - S_q| < 1$, $\forall$ $p > q \geq m$

$|S_p - S_m| < 1$, $\forall$ $p > m$

$-1 < S_p - S_m < 1$ $\forall$ $p > m$

$S_m - 1 < S_p < S_m + 1$

$k_1 = \min\{S_m - 1, S_1, S_2 \ldots S_m\}$

$k_2 = \max\{S_m + 1, S_1, S_2 \ldots S_m\}$

$k_1 \leq S_n \leq k_2$.

$\therefore \{S_n\}$ is bounded.

$\Rightarrow$ we have a bounded sequence $\{S_n\}$

By Bolzano-weistrass theorem $\exists$ a limit

point say l to $\{s_n\}$.
If there is another limit point say l' to $\{s_n\}$

$$\varepsilon = |l - l'| > 0$$

Since $\{s_n\}$ is cauchy and l, l' are limit points

$\exists\, m \in N \ni$

$$|s_p - s_q| < \frac{\varepsilon}{3}, \ \forall\, p > q \geq m$$

$$|s_p - l| < \frac{\varepsilon}{3}, \ \forall\, p \geq m$$

$$|s_q - l'| < \frac{\varepsilon}{3}, \ \forall\, q \geq m$$

$$|l - l'| = |l - s_p + s_p - s_q + s_q - l'|$$

$$\leq |l - s_p| + |s_p - s_q| + |s_q + l'|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$$|l - l'| < \varepsilon = |l - l'|$$

this is contradiction

$\therefore \{s_n\}$ has only one limit point.

Hence $\{s_n\}$ is convergent.

Pre sm Show that

$$\lim_{n \to \infty} \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2} \right] = 0$$

$$0 \leq s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}$$

$$\leq \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2} = \frac{n}{n^2}$$

$$0 \leq s_n \leq \frac{1}{n}$$

$$\lim 0 \leq \lim s_n \leq \lim \frac{1}{n}$$

$$0 \leq \lim s_n \leq 0$$

$$\lim_{n \to \infty} s_n = 0$$

Sandwich theorem:-
If $\{S_n\}, \{t_n\}, \{u_n\}$ are three sequences such that
(i) For some positive integer k, $S_n \leq u_n \leq t_n, \forall n \in N, n \geq k$
(ii) $\lim S_n = \lim t_n = l$. then $\lim u_n = l$.

proof:-
Let $\varepsilon > 0$

since $\lim S_n = l, \exists m_1 \in N \ni n \in N, n \geq m_1 \Rightarrow$
$|S_n - l| < \varepsilon \Rightarrow l - \varepsilon < S_n < l + \varepsilon$

since $\lim t_n = l, \exists m_2 \in N \ni n \in N, n \geq m_2 \Rightarrow$
$|t_n - l| < \varepsilon \Rightarrow l - \varepsilon < t_n < l + \varepsilon$

Let $m = \max\{m_1, m_2, k\}$. then $n \in N, n \geq m \Rightarrow$
$n \geq m_1, n \geq m_2, n \geq k.$
$\Rightarrow l - \varepsilon < S_n < l + \varepsilon, l - \varepsilon < t_n < l + \varepsilon, S_n \leq u_n < t_n \Rightarrow$
$l - \varepsilon < S_n \leq u_n \leq t_n < l + \varepsilon$
$\Rightarrow l - \varepsilon < u_n < l + \varepsilon \Rightarrow |u_n - l| < \varepsilon$ $\therefore u_n = l.$

$$\sum_1^\infty \frac{1}{2^n + 3^n}$$

let $u_n = \frac{1}{2^n + 3^n}, \forall n$

$= \frac{1}{3^n}\left(\frac{1}{1 + \left(\frac{2}{3}\right)^n}\right)$ Take $v_n = \frac{1}{3^n}$.

It is an I.G.P

By comparision test since $r = 3 > 1$

$$\underset{n \to \infty}{Lt} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n} = \frac{1}{1+0} = 1$$

Hence Σu_n also converges.

If $S_n = \left[1+\frac{1}{n}\right]^n$ then show that $\{S_n\}$ is convergent.

If $S_n = \left[1+\frac{1}{n}\right]^n$

By the binomial theorem.

$$S_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Similarly,

$$S_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

$$> 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)+\cdots+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$\therefore S_{n+1} > S_n$ for all $n \in N$.

Hence $\{S_n\}$ is monotonically increasing.

Also, from ①;

$$2 < S_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$$